On Painlevé analysis for some non–linear evolution equations

By

M. El-Sabbagh[(1)], R. Gad, A. Shehata[(2)], and A. Saleh[(3)].

Mathematics Department, Faculty of Science, Minia University, Egypt.

Abstract

In this paper, we present explicit Painlevé test for the potential Boussinesq equation, The murrary equation, The (2 + 1) Calogero equation, The Rosenau – Hyman equation (RH), Cole – Hopf (CH) equation, The Fornberg – Whitham equation (FW), Some of these equations have shown to possess Painlevé property, therefore, are Painleve integrable while the rest did not pass the test and reasons for that are conjectured.

Keywords: Painlevé analysis method, The potential Boussinesq equation, The murrary equation, The (2 + 1) Calogero equation, The Rosenau – Hyman equation (RH), Cole – Hopf (CH) equation, The Fornberg – Whitham equation (FW), integrability, Bäcklund transformation.

Introduction:

Nonlinear partial differential equations (NLPDEs) [1] are widely used to describe complex phenomena in various fields of sciences, especially in physics. Therefore solving nonlinear problems plays an important role in nonlinear sciences. In this direction, many effective methods for determining exact solutions of NLPDEs have been established and developed during the past few decades. Among the various different methods, the Lie symmetry method, also called Lie group method, is one of the most powerful methods to determine solutions of NLPDEs. The fundamental basis of this method is that when a differential equation is invariant under a Lie group of transformations [2–4], a reduction transformation exists. For PDEs with two independent variables, a single group reduction transforms the PDEs into ordinary differential equations (ODEs), which are generally easier to solve. In the recent past there have been considerable developments in symmetry methods for differential equations as is evident by the number of research papers, books and new symbolic software devoted to the subject.

The (2 + 1)–dimensional PKP equation [5]:

$$\sigma_{xt} + \frac{3}{2}\,\sigma_x\sigma_{xx} + \frac{1}{4}\,\sigma_{xxxx}\,\frac{3}{4}\,\sigma_{yy} = 0\,, \qquad (1.1)$$

describes the dynamics of 2–dimensional, small, but finite amplitude waves and solitons in a variety of media, for example, in plasma physics, hydrodynamics and solid–state physics. Eq. (1.1) is also derived in various physical contexts assuming that the wave is moving along x and all changes in y are slower than in the direction of motion [6]. By using various techniques and methods exact traveling wave solutions, linear solitary wave solutions, soliton–like solutions and some numerical solutions were obtained in [7–10]. However, in multifarious real physical backgrounds, nonlinear partial differential equations with variable coefficients often provide more powerful and realistic models than their constant coefficient counterparts when the inhomogeneities of media is considered. So it is of great importance to find exact solutions of NLPDEs with variable coefficients and recently, many authors have researched in this direction [11–16].

The integrability of non–linear partial differential equations (NLPDEs) is an interesting topic in non–linear sciences. Many methods have been established by mathematicians and physicists to study the integrability of NLPDEs. Some of the most important methods and notations , for integrability are the bilinear method [17], the symmetry reductions [18], Bäcklund and Darboux transformations [19], the Painleve analysis method

---

(1) Sabbagh52@hotmail.com, elsabbaghmostafa@yahoo.com.

(2) Shehata1433@yahoo.com.

(3) Ahmed_saleh1989@yahoo.com.

and the Lax pairs can be found from the Painlevé analysis [30], the Lax pairs of many Painlevé integrable models have not yet been found [25,26]. Therefore, when saying a model is integable, we must say under what specific meaning(s). for example, we say a model is Painlevé inegrable if the model has the Painlevé property, and a model is Lax or IST [27,28](inverse scattering transformation) integrable if the model has a Lax pair and can be solved by the IST approach.

Painlevé analysis method:

Painlevé property is a method of investigation for the integrability properties of many NLEEs . If a PDE which has no points such as movable branch, algebraic and logarithmic then is called P–type. An ordinary differential equation (ODE) might still admit movable essential singularities without movable branch points. This method does not identify essential singularities and therefore it provides only necessary conditions for an ODE to be of P–type. Singularity structure analysis admitting the P–property advocated by Ablowitz et al. For ODEs and extended to PDE by Weiss, Tabor and Carnevale (WTC), plays a key role of investigating the integrability properties of many NLEEs. The well–known procedure of WTC requires,

1. The determination of leading orders Laurent series,
2. The identification of powers at which the arbitrary functions can enter into the Laurent series called resonances,
3. Verifying that , at the resonance values , sufficient number of arbitrary functions exist without introducing the movable critical manifold.

According to the WTC method, the general solution of PDE is in the below from

$$u(x, t) = \varphi^{\alpha}(x, t)\sum_{j=0}^{\infty} u_j\,(x, t)\,\varphi^{j}\,(x, t), \tag{2.1}$$

where α is negative integer, $\varphi(x, t) = 0$ is the equation of singular manifold. The functions $u_j\ (j = 0, 1, 2,...)$ have to be determined by substitution of expansion (2.1) into the PDE, So PDE becomes

$$\sum_{j=0}^{\infty} E_j(u_0, ..., u_j, \varphi)\,\varphi^{j+q}\,(x, t) = 0, \tag{2.2}$$

where q is some negative constant . E_j depends on φ only by the derivatives of φ. The successive practical steps of Painlevé analysis are the following:

1– Determine the possible leading orders α by balancing two or more terms of the PDE and expressing that they dominate the other terms.

2– Solve equation $E_0 = 0$ for non–zero values of u_0 ; this may lead to several solutions, called branches.

3– Find the resonances, i.e. the values of j for which u_j cannot be determined from equation $E_j = 0$. This last equation has generally the form

$$E_j = (j + 1)\,p(j)\,\varphi_x^{i}\ \ \varphi_t^{n-i} u_j + Q\,(u_0, ..., u_{j-1}, \varphi) = 0,\ \forall\ j > 0, \tag{2.3}$$

Where n is order of the PDE, $0 \le i \le n$ and p is a polynomial of degree $n-1$.

The values of the resonances are the zeros of p .

4– Determine whether the resonances are , compatible, or not. At resonance , after substitution in (2.3) of the previously computed u_i, $i \le j - 1$, the function Q is either zero or non–zero then in the case u_j can be arbitrarily chosen and the expansion (2.2) does not exist for arbitrary φ, so the resonance is called compatible.

5– All resonances occur at positive integer values of j and are compatible.

3. The potential Boussinesq equation :

$$u_{tt} + u_x u_{xx} + u_{xxxx} = 0, \tag{3.1}$$

We first present the Painlevé test of the potential Boussinesq equation. According to the WTC method, the general solution of PDE is in the form

$$u(x, t) = \varphi^{\alpha}\,(x, t)\,\sum_{j=0}^{\infty} u_j\,(x, t)\,\varphi^{j}(x, t), \tag{3.2}$$

where α is negative , $\varphi(x, t) = 0$ is the equation of singular manifold.

The function u_j ($j = 0, 1, 2,...$) have to be determined by substitution of expansion into the PDE, So it becomes

$$\sum_{j=0}^{\infty} E_j\,(u_0 ,...,u_j ,\varphi)\, \varphi^{j+q}\,(x,t) = 0, \tag{2.2}$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.

The leading order of solution of equation (3.2) is assumed as

$$u \approx u_o\, \varphi^{\alpha} \;, \tag{3.3}$$

Substituting Eq. (3.3) into (3.1) and equating the most dominant terms,
the following results are obtained

$$\alpha = -1, \quad u_o = 12\varphi_x \,. \tag{3.4}$$

For finding the resonances, the full Laurent series :

$$u = u_0\, \varphi^{-1} + \sum_{j=1}^{\infty} u_j\, \varphi^{j-1} \;, \tag{3.5}$$

is substituted into Equation (3.1) and by equating the coefficients of φ^{j-5} , the polynomial equation in j is derivated as

$$j^3 - 4\,j^2 - j + 4 = 0,$$

$$(j - 1)(j + 1)(j - 4) = 0, \tag{3.6}$$

Using the previous Eq. (3.4), the resonances are found to be j = –1 , 1, 4

As usual, the resonance at j = –1 corresponds to the arbitrariness of singular manifold $\varphi(x , y, z, t) = 0$. In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (3.5) is substituted in Eq. (3.1). From the coefficient of φ^{-5}, the explicit value of u_0 is obtained as given in Eq. (3.4). Collecting the coefficient of φ^{-4}, the result is obtained as zero. Absence of u_1 proves that u_1 is arbitrary. This corresponds to the resonance value at j = 1. As solving these algebraic equations by Maple program , we obtain the results: u_1, u_2,u_3,u_4,u_5,u_6 . Collecting the coefficient of φ^{-3} , the following equation is obtained to give u_2 as

$$u_2 = \frac{1}{2}\frac{1}{u_0\varphi_x^3}\,(- 8\, u_0\, \varphi_x\, \varphi_{x,x,x,} + u_0\, \varphi_x\, u_{0,x,x} + u_{0,x}\, u_0\, \varphi_{x,x} + 2u_{0,x}^2\, \varphi_x - 2\, u_{1,x}\, u_0\, \varphi_x^2 - 6\, u_o\, \varphi_{x,x}^2 - 24\, u_{0,x}\, \varphi_x\, \varphi_{x,x} - 12\, u_{0,x,x}\, \varphi_x^2). \tag{3.7}$$

Proceeding further to the coefficient of φ^{-2}, the value of u_3 is obtained as

$$u_3 = \frac{1}{2}\frac{1}{u_0\varphi_x^3}\,(- 6\, u_{0,x,x}\, \varphi_{x,x} + u_0\, \varphi_{x,x,x,x} + 2\, u_0\, \varphi_x\, u_2\, \varphi_{x,x} + u_0\, \varphi_x\, u_{1,x,x} - u_{o,x}\, u_{0,x,x} + u_{1,x}\, u_o\, \varphi_{x,x} + 2\, u_{1,x}\, \varphi_x\, u_{o,x} + 2\, u_2\, u_{0,x}\, \varphi_x^2 + 4\, u_{0,x,x,x}\, \varphi_x + 4\, u_{o,x}\, \varphi_{x,x,x}). \tag{3.8}$$

Collecting the coefficient of φ^{-1} , the result is obtained as zero. Absence of u_4 proves that u_4 is arbitrary. This corresponds to the resonance value at j = 4.

And so on. we conclude that the equation be amenable to integration possible.

To construct the Bäcklund transformation of Eq. (3.1), let us truncate the Laurent series

$$u = \frac{u_0}{\varphi} + u_1 + u_2\varphi + u_3\varphi^2 + u_4\varphi^3 ,$$

Hence

$$u = \frac{12\varphi_x}{\varphi} + u_1 + u_2\, \varphi + u_3\, \varphi^2 + u_4\, \varphi^3 \,. \tag{3.7}$$

where the pair of function (u,u_4) satisfy Eq. (3.1) and hence Eq. (3.7) may be the associated Bäcklund transformation of Eq. (3.1).

4. The murrary equation :

$$u_{xx} + \lambda_1\, u\, u_x + \lambda_2\, u - \lambda_3\, u^2 - u_t = 0, \tag{4.1}$$

λ_1 , λ_2 , λ_3 are constant.

We first present the Painlevé test of the murrary equation. According to the WTC method, the general solution of PDE is in the form

$$u(x,t) = \varphi^{\alpha}(x,t) \sum_{j=0}^{\infty} u_j(x,t)\, \varphi^{j}(x,t), \tag{4.2}$$

where α is negative , and $\varphi(x, t) = 0$ is the equation of singular manifold.
The function u_j ($j = 0, 1, 2,\ldots$) have to be determined by substitution of expansion into the PDE, So it becomes

$$\sum_{j=0}^{\infty} E_j(u_0, \ldots, u_j, \varphi)\, \varphi^{j+q}(x,t) = 0,$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.
The leading order of solution of equation (4.2) is assumed as

$$u \approx u_o\, \varphi^{\alpha}. \tag{4.3}$$

Substituting Eq. (4.3) into (4.1) and equating the most dominant terms, the following results are obtained

$$\alpha = -1, \quad u_o = \frac{2\varphi_x}{\lambda 1} \tag{4.4}$$

For finding the resonances, the full Laurent series :

$$u = u_0\, \varphi^{-1} + \sum_{j=1}^{\infty} u_j\, \varphi^{j-1} \tag{4.5}$$

is substituted into Equation (4.1) and by equating the coefficients of φ^{j-3} , the polynomial equation in j is derivated as

$$(j+1)(j-2)\,\lambda_1 = 0, \tag{4.6}$$

Using the previous Eq. (4.4), the resonances are found to be j = –1 , 2.
As usual, the resonance at j = –1 corresponds to the arbitrariness of singular manifold $\varphi(x, y, z, t) = 0$. In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (4.5) is substituted in Eq. (4.1). From the coefficient of φ^{-3}, the explicit value of u_0 is obtained as given in Eq. (4.4). Collecting the coefficient of φ^{-2}, the following equation is obtained to give u_1 as solving these algebraic equations by Maple program , we obtain the results: u_1, u_2, u_3, u_4, u_5, u_6

$$\left\{ u_1 = -\frac{-u_{0,x}\,\lambda_1\, u_0 + 2\, u_{0,x}\,\phi_x + u_0\,\phi_{x,x}}{u_0\,\phi_x\,\lambda_1} \right\}$$

Collecting the coefficient of φ^{-1} , the result is obtained as zero. Absence of u_2 proves that u_2 is arbitrary. This corresponds to the resonance value at j = 2.
From the coefficient of φ^{0}, the value of u_3 is obtained as

$$u_3 = -\frac{1}{2\varphi_x + u_0\lambda_1}\left(u_2\varphi_{x,x} + 2\, u_{2,x}\,\varphi_x + u_{0,x}\,\lambda_1\, u_2 + u_{1,x,x} + u_{2,x}\,\lambda_1\, u_0 + \lambda_1\, u_1\, u_{1,x} + \lambda_1\, u_1\, u_2\,\varphi_x \right).$$

Proceeding further to the coefficient of φ^{-2}, the value of u_3 is obtained as

$$u_4 = -\frac{1}{2}\frac{1}{\varphi_x(3\varphi_x + u_0\lambda_1)}\left(\lambda_1\, u_2\, u_{1,x} + \lambda_1\, u_2^2\,\varphi_x + 2\, u_3\,\varphi_{x,x} + u_{3,x}\,\lambda_1\, u_0 + u_{0,x}\,\lambda_1\, u_3 + 4\, u_{3,x}\,\varphi_x + \lambda_1\, u_1\, u_{2,x} + 2\,\lambda_1\, u_1\, u_3\,\varphi_x + u_{2,x,x}\right).$$

and so on. we conclude that the equation be possess to integration possible.
To construct the Bäcklund transformation of Eq. (4.1), let us truncate the Laurent series

$$u = \frac{u_0}{\varphi} + u_1 + u_2\varphi + u_3\varphi^2 + u_4\varphi^3,$$

Hence

$$u = \frac{2\varphi_x}{\lambda 1\, \varphi} + u_1 + u_2\,\varphi + u_3\,\varphi^2 + u_4\,\varphi^3. \tag{4.7}$$

where the pair of function (u, u_4) satisfy Eq. (4.1) and hence Eq. (4.7) may be the associated Bäcklund transformation of Eq. (4.1).

5. The (2 + 1) Calogero equation

$$u_{xxxy} - 2\, u_y\, u_{xx} - 4\, u_x\, u_{xy} + u_{xt} = 0, \tag{5.1}$$

We first present the Painlevé test of the Calogero equation. According to the WTC method, the general solution of PDE is in the form

$$u(x, t) = \varphi^{\alpha}(x, t) \sum_{j=0}^{\infty} u_j(x, t)\, \varphi^{j}(x, t), \quad (5.2)$$

where α is negative , $\varphi(x, t) = 0$ is the equation of singular manifold.

The function u_j $(j = 0, 1, 2,...)$ have to be determined by substitution of expansion into the PDE, So it becomes

$$\sum_{j=0}^{\infty} E_j(u_0, ..., u_j, \varphi)\, \varphi^{j+q}(x,t) = 0,$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.

The leading order of solution of equation (5.2) is assumed as

$$u \approx u_o\, \varphi^{\alpha}. \quad (5.3)$$

Substituting Eq. (5.3) into (5.1) and equating the most dominant terms,

and so on.the following results are obtained

$$\alpha = -1\ ,\ u_o = -\,12\varphi_x\,. \quad (5.4)$$

For finding the resonances, the full Laurent series :

$$u = u_0\, \varphi^{-1} + \sum_{j=1}^{\infty} u_j\, \varphi^{j-1}, \quad (5.5)$$

is substituted into Equation (5.1) and by equating the coefficients of φ^{j-5} , the polynomial equation in j is derivated as

$$j^2 - 5j - 6 = 0,$$

$$(j + 1)(j - 6) = 0. \quad (5.6)$$

Using the previous Eq. (5.4), the resonances are found to be j = –1 , 6

As usual, the resonance at j = –1 corresponds to the arbitrariness of singular manifold $\varphi(x, y, z, t) = 0$. In order to check the existence of sufficient number of arbitrary functions at the other resonance values, the full Laurent expansion (5.5) is substituted in Eq. (5.1). From the coefficient of φ^{-5}, the explicit value of u_0 is obtained as given in Eq. (5.4). Collecting the coefficient of φ^{-4}, the result is obtained as zero. Absence of u_1 proves that u_1 is arbitrary. This corresponds to the resonance value at j = 6. As solving these algebraic equations by Maple program , we obtain the results: u_1, u_2,u_3,u_4,u_5,u_6 Collecting the coefficient of φ^{-3}, the following equation is obtained to give u_1 as solving these algebraic equations by Maple program , we obtain the results: u_1, u_2,u_3,u_4,u_5,u_6

we conclude that the equation be satisfy to integration possible.

To construct the Bäcklund transformation of Eq. (5.1), let us truncate the Laurent series

$$u = \frac{u_0}{\varphi} + u_1 + u_2\varphi + u_3\varphi^2 + u_4\varphi^3,$$

Hence

$$u = \frac{-\,12\varphi_x}{\varphi} + u_1 + u_2\,\varphi + u_3\,\varphi^2 + u_4\,\varphi^3\,. \quad (5.7)$$

where the pair of function (u,u_4) satisfy Eq. (5.1) and hence Eq. (5.7) may be the associated Bäcklund transformation of Eq. (5.1), relating a solution u with a known solution u_1 of the Eq. (5.1) which can be taken to be a known solution.

6. The Rosenau – Hyman equation (RH) :

$$u\, u_x + 3\, u_x\, u_{xx} + u\, u_{xxx} - u_t = 0, \quad (6.1)$$

We first present the Painlevé test of the Rosenau – Hyman equation. According to the WTC method, the general solution of PDE is in the form

$$u(x, t) = \varphi^{\alpha}(x, t) \sum_{j=0}^{\infty} u_j(x, t)\, \varphi^{j}(x, t), \quad (6.2)$$

where α is negative , $\varphi(x, t) = 0$ is the equation of singular manifold. The function u_j $(j = 0, 1, 2,...)$ have to be determined by substitution of expansion into the PDE, So it becomes

$$\sum_{j=0}^{\infty} E_j(u_0, ..., u_j, \varphi)\, \varphi^{j+q}(x,t) = 0. \quad (6.3)$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.
The leading order of solution of equation (6.2) is assumed as

$$u \approx u_o \varphi^{\alpha} . \quad (6.4)$$

Substituting Eq. (6.4) into (6.1) and equating the most dominant terms,
the wanted α and u_0 results could not be computed because balancing (comparison) two or more terms of the PDE leads to a contradiction as

$$2\alpha+1= 2\alpha-3$$
$$0 = -4 \qquad \text{contradiction}$$

So It is not Painlevé integrable.
We note that the Rosenau – Hyman equations (RH) has no (does not contain) linear higher order derivative term.

7. Cole – Hopf (CH) equation :

$$u\, u_{xt} - u_x u_t - u_x^2 = 0 , \quad (7.1)$$

We first present the Painlevé test of the A Cole – Hopf (CH) equation. According to the WTC method, the general solution of PDE is in the form

$$u(x, t) = \varphi^{\alpha} (x, t) \sum_{j=0}^{\infty} u_j (x, t)\, \varphi^j(x, t). \quad (7.2)$$

where α is negative , $\varphi(x, t) = 0$ is the equation of singular manifold. The function u_j $(j = 0, 1, 2,...)$ have to be determined by substitution of expansion into the PDE, So it becomes

$$\sum_{j=0}^{\infty} E_j (u_0 , ...,u_j ,\varphi)\, \varphi^{j+q} (x,t) = 0. \quad (7.3)$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.
The leading order of solution of equation (7.2) is assumed as

$$u \approx u_o \varphi^{\alpha}. \quad (7.4)$$

Substituting Eq. (7.4) into (7.1) and equating the most dominant terms,

the following results are not obtained because balancing two or more terms of the PDE more terms of the PDE leads to a contradiction as

$$2\alpha-2= 2\alpha-2$$
$$0 = 0 \qquad \text{contradiction}$$

So It is not Painlevé integrable.
We note that the Cole – Hopf (CH) equation has no (does not contain) linear higher order derivative term.

8.The Fornberg – Whitham equation (FW) :

$$u_t - u_{xxt} + u_x - u\, u_{xxx} + u\, u_x - 3\, u_x u_{xx} = 0, \quad (8.1)$$

We first present the Painlevé test of the Fornberg – Whitham (FW) equation. According to the WTC method, the general solution of PDE is in the form

$$u(x, t) = \varphi^{\alpha} (x, t) \sum_{j=0}^{\infty} u_j (x, t)\, \varphi^j(x, t) . \quad (8.2)$$

where α is negative , $\varphi(x, t) = 0$ is the equation of singular manifold. The function u_j $(j = 0, 1, 2,...)$ have to be determined by substitution of expansion into the PDE, So it becomes

$$\sum_{j=0}^{\infty} E_j (u_0 , ...,u_j ,\varphi)\, \varphi^{j+q} (x,t) = 0. \quad (8.3)$$

where q is some negative constant. E_j depends on φ only by the derivatives of φ.
The leading order of solution of equation (8.2) is assumed as

$$u \approx u_o \varphi^{\alpha}. \quad (8.4)$$

Substituting Eq. (8.3) into (8.1) and equating the most dominant terms, the wanted α and u_0 results are could not be computed because balancing (comparison) two or more terms of the PDE leads to a contradiction as

$$2\alpha-1=2\alpha-3$$

$$0=-2 \qquad \text{contradiction}$$

So It is not Painlevé integrable.
We note that for the Fornberg – Whitham (FW) equation the linear higher order derivative term and the non-linear term are of the same derivative orders .

References :

[1] W.F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1972.
[2] P.J. Olver, Applications of Lie groups to differential equations, Graduate Texts Math., Springer–Verlag, New York, 1993.
[3] G.W. Bluman, J.D. Cole, Similarity Methods for Differential Equations, Springer–Verlag, New York, 1974.
[4] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.
[5] V.V. Kadomstev, V.I. Petviashvili, On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl. 15 (1970) 539–541.
[6] M.J. Ablowitz, P.A. Clarkson, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1990.
[7] D. Kaya, S.M. El–Sayed, Numerical soliton–like solutions of the potential Kadomstev–Petviashvili equation by the decomposition method, Phys. Lett. A320 (2003) 192–199.
[8] Z. Dai, J. Liu, Z. Liu, Exact periodic kink–wave and degenerative soliton solutions for potential Kadomstev–Petviashvili equation, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2331–2336.
[9] D. David, N. Kamran, D. Levi, P. Winternitz, Symmetry reduction for the Kadomstev–Petviashvili equation using a loop algebra, J. Math. Phys. 27 (1986) 1225–1237.
[10] D. Li, H. Zhang, New soliton–like solutions to the potential Kadomstev–Petviashvili (PKP) equation, Appl. Math. Comput. 146 (2003) 381–384.
[11] R.K. Gupta, S. Kumar, K. Singh, Benjamin–Bona–Mahony (BBM) equation with variable coefficients: similarity reductions and Painlevé analysis, Appl. Math. Comput. 217 (2011) 7021–7027.
[12] A. Bansal, R.K. Gupta, Lie Point Symmetries and Similarity Solutions of the Time–Dependent Coefficients Calogero–Degasperis Equation, Phys. Scr. 86 (2012) 035005 (11 pp.).
[13] M.S. Bruzn, M.L. Gandarias, Classical and nonclassical symmetries for the Krichever–Novikov equation, Theor. Math. Phys. 168 (2011) 875–885.
[14] R.K. Gupta, K. Singh, Symmetry analysis and some exact solutions of cylindrically symmetric null fields in general relativity, Commun. Nonlinear Sci.Numer. Simul. 16 (2011) 4189–4196.
[15] A. Guo, J. Lin, Exact solutions of (2+1)–dimensional HNLS equation, Commun. Theor. Phys. 54 (2010) 401–406.
[16] K. Singh, R.K. Gupta, Lie symmetries and exact solutions of a new generalized Hirota–Satsuma coupled KdV system with variable coefficients, Int. J. Eng. Sci. 44 (2006) 241–255.
[17] R. Hirota, Exact solutions of the Korteweg–de Vries equation for multiple collisions of solitons, phys. Rev. Lett. 27 (1971) 1192–1194.
[18] P. J. Olver, Application of Lie Group to Differential Equation, 2nd ed, Graduate Texts Math. 107, Springer, New York 1993.
[19] C–h. Gu, IDMP preprint 9402 (1994), Lett. Math. Phys. 199 (1992).
[20] F. Calogero and M. C. Nucci, Lax pairs galore, J. Math. Phys. 32 (1991) 72–74.
[21] M. F. El–Sabbagh and A. H. Khater, The painlevé property, complete integrability and Bäcklund transformations for a family of Liouvile equations, Proc. Pakistan Acad. Sci. 26 (1989) 1–5.

[22] M. F. El–Sabbagh, On the painlevé propert for the class of AKNS–Zakharov and Shabat nonlinear evolution equations, IlNuovo Cimento 101 (1988) 697–702.
[23] J. Weiss, M. Tabor, and G. Carnevale, The painlevé property for partial differential equations, J. Math. phys. 24 (1983) 522–526.
[24] J. Weiss, The singular manifold methods. In: Levi D, Winternitz P, editors. Painlevé transcedents, their asymptotics and physical applications. NATO advanced science institutes series B: physics, vol. 278. New York: plenum Press, (1992) 225–247.
[25] S.Y. Lou, Searching for higher dimensional integrable models from lower ones via painlevé analysis, phys. Rev. Lett. 80 (1998) 5027–5031.
[26] S.Y. Lou and J.J. Xu, Higher dimensional painlevé integrable method from the Kadomtsev – petviashvili, J. Math. Phys. 39 (1998) 5364–5376.
[27] M.J. Ablowitz and P.A. Clarkson, Soliton, nonlinear evolution equations and inverse scattering, New York: Cambridge University (1991).
[28] V.O. Vakhnenko, E.J. Parkes and A.J. Morrison, A Bäcklund transformation and the inverse scattering transform method for the generalized Vakhnenko equation. Chaos, Solitons & Fractals 17 (2003) 683–692.
[29] S. Y. Lou, X. y. Tang, Q. P. Liu and T. Fukuyama, Second order Lax pairs of nonlinear partial differential equations with Schwarz variants, Nlin. SI/0108045 (2001).
[30] Nikolai A. Kudryashov, Seven common errors in finding exact solutions of nonlinear differential quations, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 3507–3529.
[31] A. (Kalkanlı) Karasu, "Painlevé classification of coupled Korteweg–de Vries systems," Journal of Mathematical Physics, vol. 38, no. 7, pp. 3616–3622, 1997.
[32] S. Yu. Sakovich, "Coupled KdV equations of Hirota–Satsuma type," Journal of Nonlinear Mathematical Physics, vol. 6, no. 3, pp. 255–262, 1999 [arXiv: solv-int/9901005].
[33] R. Hirota, X. B. Hu, and X. Y. Tang, "A vector potential KdV equation and vector Ito equation: soliton solutions, bilinear Bäcklund transformations and Lax pairs," Journal of Mathematical Analysis and Applications, vol. 288, no.1, pp. 326–348, 2003.
[34] Ramani A., Grammaticos B. Tremblay S., Integrable systems without the Painlevé property, J. Phys. A: Math. Gen. 33 (2000), 3045-3052.
[35] Weiss J., The Painlevé property for partial differential equations. II. Bäcklund transformations, Lax pairs, and the Schwarzian derivative, *J. Math. Phys.* 24 (1983), 1405–1413.
[36] Steeb W.-H., Kloke M., Spieker B.M., Liouville equation, Painlevé property and Bäcklund transformation, *Z. Naturforsch. A* 38 (1983), 1054–1055.
[37] Musette M., Conte R., Algorithmic method for deriving Lax pairs from the invariant Painlevé analysis of nonlinear partial differential equations, *J. Math. Phys.* 32 (1991), 1450–1457.
[38] Karasu-Kalkanlı A., Sakovich S.Yu., Bäcklund transformation and special solutions for the Drinfeld–Sokolov–Satsuma–Hirota system of coupled equations, *J. Phys. A: Math. Gen.* 34 (2001), 7355–7358,nlin.SI/0102001.
[39] Karasu-Kalkanlı A., Karasu A., Sakovich A., Sakovich S., Turhan R., A new integrable generalization of the Korteweg–de Vries equation, *J. Math. Phys.* 49 (2008), 073516, 10 pages, arXiv:0708.3247.
[40] Conte R., Musette M., The Painlevé handbook, Springer, Dordrecht, 2008.
[41] Hone A.N.W., Painlevé tests, singularity structure and integrability, in Integrability, Editor A.V. Mikhailov, *Lecture Notes in Physics*, Vol. 767, Springer, Berlin, 2009, 245–277, nlin.SI/0502017.